International Journal of Theoretical Physics, Vol. 25, No. 9, 1986

$SU(2,2|N) \cdot \langle N' \rangle$ Superunified Theory

Shao Changgui,[1] Xu Bangqing,[2] and Guo Youzhong[3]

Received May 14, 1986

We establish a superspace U for supergauge actions, a conformal supergroup $SU(2,2|N)$, and a conformal extended $SU(2,2|N)$ supergravity theory. Using the Lagrangian Higgs evolution mechanism under the supergroup $SU(2,2|N) \otimes SU(N)$ acting on the superspace U, we advance an $SU(2,2|N) \otimes SU\langle N' \rangle$ superunified theory of a superunited system, discuss the Lagrangian evolution of the superunified theory, and give the fiber bundle geometry of all the above mechanisms.

1. INTRODUCTION

Conformal supergravities are the supersymmetric extensions of Weyl's gravity. There are different classes of conformal supergravities (Kaku *et al.*, 1977a,b, 1978; Ferrara *et al.*, 1977, 1978; Bergshoeff et al, 1980). The gauge superalgebras of these theories can be obtained by using the graded conformal groups (Kaku et al., 1977a,b; Ferrara et al., 1977; Bergshoeff et al., 1980), or by using the auxiliary fields (Ferrara et al., 1978). The internal groups may be $U(1)$ (Kaku et al., 1977a) or $SU(N)$ (Kaku et al., 1977a; Ferrara et al., 1977; Bergshoeff et al., 1980).

In this paper the internal group is chosen as $SU(N)$, but its generators are different from those of other authors (Ferrara et al., 1977; Kaku et al., 1977b; Beyl, 1979). Thus the structure constants of supergroup $SU(2,2|N)$ are different, and the couplings are different from those of other theories. As mentioned below, the $SU(2,2|N)$ generators we give are simple and some grand unified theories (GUTs) can be imbeded into our theory.

In this paper we define group $SU(2,2|N) \cdot (N) \equiv SU(2,2|N) \otimes SU(N)$, $SU(2,2|N) \cdot \langle N' \rangle \equiv SU(2,2|N) \otimes SU\langle N' \rangle$. The group $SU\langle N' \rangle$ is defined in Section 4.

[1]Wuhan Institute of Technology, China.
[2]Institute of Atomic Energy, Academia Sinica, China.
[3]Wuhan Institute of Mathematical Sciences, Academia Sinica, China.

0020-7748/86/0900-0945$05.00/0

2. SUPERSPACE FOR SUPERGAUGE ACTIONS OF A SUPERUNIFIED SYSTEM AND ITS ORIGINAL LAGRANGIAN

Let us denote the space-time manifold by M [the signature of M is $(1, 1, 1, -1)$]. Take a superunified system which includes space-time, and suppose that the gravitational masses, particles, and fields are also present in the system. The three kinds of gauge symmetries of the system, i.e., the space-time symmetries, the internal symmetries, and the supersymmetry, are represented by the supergroup G. Thus $\forall x \in M$, $\exists a$ superspace spanned by the sets of the physical and mathematical quantities under G. We denote the superspace by U. The superspace U can be chosen in the form

$$U = V \oplus_G H \oplus \Omega_{Gi}$$

where V is the associated Bose space for the guage action of the gauge theory of gravity; its signature is (1, 1, 1, −1, −1, 1); H is the Fermi superspace of dimension $n \times m$, and the supervector in H is $\{\mathcal{O}^i_\alpha\}$ $(i=1, 2, \ldots, n;\ \alpha = 1, 2, \ldots m)$, where $\{\mathcal{O}^i_\alpha\}$ is a set of physical fields on M; Ω is the Bose Higgs space for the representation of the internal symmetry group G_i; and $V \oplus_G H$ is the representation space of the supergroup G of supergravity; in this paper G is chosen as $SU(2,2|N)$; $\oplus$ is the direct sum.

Thus every $\{\mathcal{O}^i_\alpha\} \in H$ is a supervector in H and on M. Let us use the Majorana index; then $m=4$ and H is a Fermi–Bose space of dimension $n \times 4$. In this case the $\{\mathcal{O}^i_\alpha\} \in H$ is a set of Fermi fields (or multiplets) constructed by quarks and leptons on M and these Fermi fields can be transformed with the supersymmetry transformations.

The original Lagrangian of the system can be naturally obtained from the superspace U as follows:

$$\mathcal{L}_{\text{orig}} = \mathcal{L}_m + \mathcal{L}_{\text{F}} + \mathcal{L}_{\text{B}}$$

where $\mathcal{L}_m$ is the Lagrangian of gravitational masses in M associated with V, $\mathcal{L}_{\text{F}}$ is the free Lagrangian of Fermi fields in H, and $\mathcal{L}_{\text{B}}$ is the free Lagrangian of Higgs fields in Ω.

3. AN $SU(2,2|N)$ EXTENDED SUPERGRAVITY THEORY

Here a superalgebra $su(2,2|N)$ is constructed and a supergroup $SU(2,2|N)$ is then obtained. The groups G and G_i are chosen to be the groups $SU(2,2|N)$ and $SU(N)$, respectively, in this case, and an extended conformal supergravity theory which takes $SU(N)$ as the gauge group of its GUT can thus be constructed. In this case U has the form

$$U \equiv V \oplus_{SU(2,2|N)} H \oplus \Omega_{SU(N)}$$

here the metric of V is $\eta_{\sigma\tau} = \text{diag}(1, 1, 1, -1, -1, 1)$.

It is well known that if we take the Majorana representation of Dirac matrices in Minkowski space, we have

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}, \qquad \eta_{ab} = \mathrm{diag}(1, 1, 1, -1)$$

$$\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0$$

Using these expressions, we can define the generators of a matrix representation of the conformal group $SU(2,2)$ as

$$\hat{C} = \left(\begin{array}{c|c} M_{ab}, P_a, K_a, D & 0 \\ \hline 0 & 0 \end{array}\right)$$

where

$$M_{ab} = \sigma_{ab}, \qquad \sigma_{ab} = \tfrac{1}{4}[\gamma_a, \gamma_b]$$

$$P_a = \tfrac{1}{2}\gamma_a, \qquad K_a = \tfrac{1}{2}\gamma_a\gamma_5, \qquad D = \tfrac{1}{2}\gamma_5$$

and

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

In constructing the above expressions, we have taken $\lambda = 1$ for the de Sitter curvature, and made the algebra $ds(3,2)$ a subalgebra of the conformal algebra $su(2,2)$ of (M_{ab}, P_a). The $ds(3,2)$ subalgebra will enable us to obtain a Yang-Mills type of action, which is invariant under the Weyl mapping. Let us take the generators of internal symmetry groups as

$$\bar{I} = (A, E_k)$$

where

$$A = -\frac{i}{4}\left(\begin{array}{c|c} I & 0 \\ \hline 0 & (4/N)I \end{array}\right), \qquad E_k = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & g_k \end{array}\right)$$

where g_k are $SU(N)$ generators. We select the supersymmetry generators as

$$S^i_\alpha = \left(\begin{array}{c|c} 0 & 0 \quad L_{(i)\alpha} \quad 0 \\ \hline 0 & \\ (CR)_{\alpha(i)} & 0 \\ 0 & \end{array}\right)$$

$$
= \left(\begin{array}{c|c}
0 & 0\cdots0\;\overbrace{\begin{matrix} L_{1\alpha} \\ L_{2\alpha} \\ L_{3\alpha} \\ L_{4\alpha} \end{matrix}}^{i}\;0\cdots0 \\ \hline
\begin{matrix} i\left\{\begin{matrix} 0 \\ \vdots \\ 0 \\ (CR)_{\alpha 1}\,(CR)_{\alpha 2}\,(CR)_{\alpha 3}\,(CR)_{\alpha 4} \end{matrix}\right. \\ 0 \\ \vdots \\ 0 \end{matrix} & 0
\end{array}\right)
$$

$$
Q^i_\alpha = \left(\begin{array}{c|c}
0 & 0 \quad R_{(i)\alpha} \quad 0 \\ \hline
\begin{matrix} 0 \\ (CL)_{\alpha(i)} \\ 0 \end{matrix} & 0
\end{array}\right)
$$

$$
= \left(\begin{array}{c|c}
0 & 0\cdots0\;\overbrace{\begin{matrix} R_{1\alpha} \\ R_{2\alpha} \\ R_{3\alpha} \\ R_{4\alpha} \end{matrix}}^{i}\;0\cdots0 \\ \hline
\begin{matrix} i\left\{\begin{matrix} 0 \\ \vdots \\ 0 \\ (CL)_{\alpha 1}\,(CL)_{\alpha 2}\,(CL)_{\alpha 3}\,(CL)_{\alpha 4} \end{matrix}\right. \\ 0 \\ \vdots \\ 0 \end{matrix} & 0
\end{array}\right)
$$

where $L=\frac{1}{2}(1-\gamma_5)$, $R=\frac{1}{2}(1+\gamma_5)$, $C=\gamma_0$, the charge conjugate matrix, and (i) runs 1, 2, 3, 4 columns (rows) at its located row (column).

The commutation relations of the Bose generators of space-time can be easily calculated as follows:

$$[M_{ab}, M_{cd}] = \eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}$$

$$[M_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b$$
$$[M_{ab}, K_c] = \eta_{bc}K_a - \eta_{ac}K_b$$
$$[P_a, P_b] = M_{ab}$$
$$[K_a, K_b] = -M_{ab}$$
$$[P_a, D] = K_a$$
$$[K_a, D] = P_a$$
$$[P_a, K_b] = \eta_{ab}D$$

We can also find other commutation and anticommutation relations of the Bose and Fermi generators of the gauge supergroup $SU(2,2|N)$ as

$$[S^i_\alpha, M_{ab}] = -(\sigma^T_{ab})_{\alpha\beta}S^i_\beta$$
$$[Q^i_\alpha, M_{ab}] = -(\sigma^T_{ab})_{\alpha\beta}Q^i_\beta$$
$$[S^i_\alpha, P_a] = -\tfrac{1}{2}(\gamma^T_a)_{\alpha\beta}Q^i_\beta$$
$$[S^i_\alpha, K_a] = \tfrac{1}{2}(\gamma^T_a)_{\alpha\beta}Q^i_\beta$$
$$[Q^i_\alpha, P_a] = -\tfrac{1}{2}(\gamma^T_a)_{\alpha\beta}S^i_\beta$$
$$[Q^i_\alpha, K_a] = -\tfrac{1}{2}(\gamma^T_a)_{\alpha\beta}S^i_\beta$$
$$[S^i_\alpha, D] = \tfrac{1}{2}S^i_\alpha$$
$$[Q^i_\alpha, D] = -\tfrac{1}{2}Q^i_\alpha$$
$$[S^i_\alpha, A] = -\frac{i}{4}\left(\frac{4}{N}-1\right)(\gamma_5)_{\alpha\beta}S^i_\beta$$
$$[Q^i_\alpha, A] = \frac{i}{4}\left(\frac{4}{N}-1\right)(\gamma_5)_{\alpha\beta}Q^i_\beta$$
$$[S^i_\alpha, (E^s_k; E^A_k)] = ((\gamma_5)_{\alpha\beta}(g^s_k)^{ij}S^j_\beta; (g^A_k)^{ij}S^j_\alpha)$$
$$[Q^i_\alpha, (E^s_k; E^A_k)] = (-(\gamma_5)_{\alpha\beta}(g^s_k)^{ij}Q^j_\beta; (g^A_k)^{ij}Q^j_\alpha)$$
$$[E_j, E_k] = if^l_{jk}E_l$$
$$\{S^i_\alpha, S^j_\beta\} = -\tfrac{1}{2}\,\delta^{ij}(\gamma^a C)_{\alpha\beta}(P_a - K_a)$$
$$\{Q^i_\alpha, Q^j_\beta\} = -\tfrac{1}{2}\delta^{ij}(\gamma^a C)_{\alpha\beta}(P_a + K_a)$$
$$\{Q^i_\alpha, S^j_\beta\} = [-\tfrac{1}{2}C_{\alpha\beta}D - (C\sigma^{ab})_{\alpha\beta}M_{ab} + (i\gamma_5 C)_{\alpha\beta}A]\delta^{ij} + \tfrac{1}{2}(\gamma_5 C)_{\alpha\beta}(g^s_k)^{ij}E^s_k + \tfrac{1}{2}C_{\alpha\beta}(g^A_k)^{ij}B^A_k, \qquad a > b$$

In these expressions T denotes the transpose. In the calculation of the above superalgebraic relations we used the following extension of Gell-Mann matrices:

$$g_k = (g_k^A, g_k^s)$$

$$g_k^A = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & & -i & \vdots \\ \vdots & \iddots & & \vdots \\ \vdots & i & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}$$

$$g_k^s = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & & 1 & \vdots \\ \vdots & \iddots & & \vdots \\ \vdots & 1 & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}, \quad \left[\frac{2}{N(N-1)}\right]^{1/2} \begin{pmatrix} i & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & i(1-N) & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}$$

The g_k are regarded as $SU(N)$ generators as mentioned above, and are different from the generators used in Ferrara et al. (1977) and Kaku et al. (1977b). The generators of the supergroup $SU(2,2|N)$ defined above can ensure that they can become Hermitian and in agreement with the gauge group generators of some GUTs (Langacker, 1981). Therefore the GUTs can be imbedded into our supergravity theory (Shao Changgui, unpublished).

Now we introduce the supergauge potential

$$B_\mu^{AB} = (B_\mu^{ab}, V_\mu^a, C_\mu^a, \varepsilon_\mu; E_\mu^k, A_\mu; \bar{\phi}_{\mu i}^\alpha, \bar{\psi}_{\mu i}^\alpha)$$

corresponding to the $SU(2,2|N)$ supergroup generators

$$\tau_{AB} = (M_{ab}, P_a, K_a, D; E_k, A; S_\alpha^i, Q_\alpha^i)$$

respectively, into our theory. Defining the supergauge covariant derivative

$$D_\mu = \partial_\mu + B_\mu^{AB}\tau_{AB}$$

We can get the formula of the supergauge field strength $R_{\mu\nu}^{AB}$ as

$$R_{\mu\nu}^{AB} = \partial_\mu B_\nu^{AB} - \partial_\nu B_\mu^{AB} + F_{CD,EF}^{AB} B_\mu^{CD} B_\nu^{EF}$$

where $F_{CD,EF}^{AB}$ are structure constants of the supergroup $SU(2,2|N)$ and can be calculated from the above superalgebraic relations. Thus we can write the supergauge field strength components corresponding to the generators τ_{AB} as follows:

$$R^{ab}_{\mu\nu}(M)=R^{ab}_{\mu\nu}(M)_{\text{gra}}+R^{ab}_{\mu\nu}(M)_{\text{sup}}$$
$$=F^{ab}_{\mu\nu}+V^{ab}_{\mu\nu}-C^{ab}_{\mu\nu}-\bar{\psi}_{\mu i}(C\sigma^{ab}C^{-1})\phi_{\nu i}-\bar{\phi}_{\mu i}(C\sigma^{ab}C^{-1})\psi_{\nu i}$$
$$R^{a}_{\mu\nu}(P)R^{a}_{\mu\nu}(P)_{\text{gra}}+R^{a}_{\mu\nu}(P)_{\text{sup}}$$
$$=-J^{a}_{\mu\beta}+C^{a}_{\mu\nu}-\tfrac{1}{2}(\bar{\phi}_{\mu i}\gamma^{a}\phi_{\nu i}+\bar{\psi}_{\mu i}\gamma^{a}\psi_{\nu i})$$
$$R^{a}_{\mu\nu}(K)=R^{a}_{\mu\nu}(K)_{\text{gra}}+R^{a}_{\mu\nu}(K)_{\text{sup}}$$
$$=K^{a}_{\mu\nu}+V^{a}_{\mu\nu}-\tfrac{1}{2}(\bar{\psi}_{\mu i}\gamma^{a}\psi_{\nu i}-\bar{\phi}_{\mu i}\gamma^{a}\phi_{\nu i})$$
$$R_{\mu\nu}(D)=R_{\mu\nu}(D)_{\text{gra}}+R_{\mu\nu}(D)_{\text{sup}}$$
$$=\varepsilon_{\mu\nu}-C_{\mu\nu}-\tfrac{1}{2}(\bar{\psi}_{\mu i}\phi_{\nu i}+\bar{\phi}_{\mu i}\psi_{\nu i})$$
$$R_{\mu\nu}(A)=R_{\mu\nu}(A)_{\text{gra}}+R_{\mu\nu}(A)_{\text{sup}}$$
$$=A_{\mu\nu}+i(\bar{\psi}_{\mu j}\gamma_5\phi_{\nu j}+\bar{\phi}_{\mu j}\gamma_5\psi_{\nu j})$$
$$R_{\mu\nu i}(S^{i})=\bar{\phi}_{\nu i}\overleftarrow{D}^{1}_{\mu}-\bar{\phi}_{\mu i}D\overleftarrow{D}^{1}_{\nu}-\tfrac{1}{2}(\bar{\psi}_{\mu i}\gamma^{T}_{\nu}-\bar{\psi}_{\nu i}\gamma^{T}_{\mu})-\tfrac{1}{2}(\bar{\psi}_{\mu i}\tilde{\gamma}^{T}_{\nu}-\bar{\psi}_{\nu i}\tilde{\gamma}^{T}_{\mu})$$
$$+(E^{A})^{k}_{\mu}(g^{A}_{k})_{ij}\bar{\phi}_{\nu j}-(E^{A})^{k}_{\nu}(g^{A}_{k})_{ij}\bar{\phi}_{\mu j}+(E^{s})^{k}_{\mu}(g^{s}_{k})_{ij}\gamma_5\bar{\phi}_{\nu j}$$
$$-(E^{s})^{k}_{\nu}(g^{s}_{k})_{ij}\gamma_5\bar{\phi}_{\mu j}$$
$$R_{\mu\nu i}(Q^{i})=\bar{\psi}_{\nu i}\overleftarrow{D}^{2}_{\mu}-\bar{\psi}_{\mu i}\overleftarrow{D}^{2}_{\nu}-\tfrac{1}{2}(\bar{\phi}_{\mu i}\gamma^{T}_{\nu}-\bar{\phi}_{\nu i}\gamma^{T}_{\mu})+\tfrac{1}{2}(\bar{\phi}_{\mu i}\tilde{\gamma}^{T}_{\nu}-\bar{\phi}_{\nu i}\tilde{\gamma}^{T}_{\mu})$$
$$+(E^{A})^{k}_{\mu}(g^{A}_{k})_{ij}\bar{\psi}_{\nu j}-(E^{A})^{k}_{\nu}(g^{A}_{k})_{ij}\bar{\psi}_{\mu j}-(E^{s})^{k}_{\mu}(g^{s}_{k})_{ij}\gamma_5\bar{\psi}_{\nu j}$$
$$+(E^{s})^{k}_{\nu}(g^{s}_{k})_{ij}\gamma_5\bar{\psi}_{\mu j}$$
$$R^{k}_{\mu\nu}(E_k)=R_{\mu\nu}(E_k)_{\text{inal}}+R_{\mu\nu}(E_k)_{\text{sup}}=(R_{\mu\nu}(E^{s}_{k}),\ R_{\mu\nu}(E^{A}_{k}))$$

In the calculation of the above expressions the following expressions have also been established:

$$F^{ab}_{\mu\nu}=\partial_\mu B^{ab}_{\nu}-\partial_\nu B^{ab}_{\mu}+B^{a}_{\mu c}B^{cb}_{\nu}-B^{a}_{\nu c}B^{cb}_{\mu}$$
$$V^{ab}_{\mu\nu}=V^{a}_{\mu}V^{b}_{\nu}-V^{a}_{\nu}V^{b}_{\mu}$$
$$C^{ab}_{\mu\nu}=C^{a}_{\mu}C^{b}_{\nu}-C^{a}_{\nu}C^{b}_{\mu}$$
$$J^{a}_{\mu\nu}=\partial_\mu V^{a}_{\nu}-\partial_\nu V^{a}_{\mu}+B^{a}_{\mu b}V^{b}_{\nu}-B^{a}_{\nu b}V^{b}_{\mu}$$
$$K^{a}_{\mu\nu}=\partial_\mu C^{a}_{\nu}-\partial_\nu C^{a}_{\mu}+B^{a}_{\mu b}C^{b}_{\nu}-B^{a}_{\nu b}C^{b}_{\mu}$$
$$V_{\mu\nu}=V^{a}_{\mu}\partial_\nu-V^{a}_{\nu}\partial_\mu$$
$$C^{a}_{\mu\nu}=C^{a}_{\mu}\partial_\nu-C^{a}_{\nu}\partial_\mu$$
$$\varepsilon_{\mu\nu}=\partial_\mu\varepsilon_\nu-\partial_\nu\varepsilon_\mu$$

$$C_{\mu\nu} = C^a_\mu V_{\nu a} - C^a_\nu V_{\mu a}$$

$$D^1_\mu \equiv \partial_\mu + B^{ab}_\mu(\sigma^T_{ab}) + \tfrac{3}{4} i\gamma_5 A_\mu - \tfrac{1}{2}\varepsilon_\mu$$

$$D^2_\mu \equiv \partial_\mu + B^{ab}_\mu(\sigma^T_{ab}) - \tfrac{3}{4} i\gamma_5 A_\mu + \tfrac{1}{2}\varepsilon_\mu$$

$$A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\gamma^T_\mu \equiv \gamma^T_a V^a_\mu$$

$$\tilde{\gamma} \equiv \gamma^T_a C^a_\mu$$

$$\begin{aligned} R_{\mu\nu}(E^s_k) &= R_{\mu\nu}(E^s_k)_{\text{inal}} + R_{\mu\nu}(E^s_k)_{\text{sup}} \\ &= \partial_\mu(E^s)^k_\nu - \partial_\nu(E^s)^k_\mu + if^k_{jl}(E^s)^j_\mu(E^s)^l_\nu \\ &\quad + \tfrac{1}{2}\bar{\psi}_{\mu i}(g^s_k)_{ij}\gamma_5\phi_{\nu j} + \tfrac{1}{2}\bar{\phi}_{\mu i}(g^s_k)_{ij}\gamma_5\psi_{\nu j} \end{aligned}$$

$$\begin{aligned} R_{\mu\nu}(E^A_k) &= R_{\mu\nu}(E^A_k)_{\text{inal}} + R_{\mu\nu}(E^A_k)_{\text{sup}} \\ &= \partial_\mu(E^A)^k_\nu - \partial_\nu(E^A)^k_\mu + if^k_{jl}(E^A)^j_\mu(E^A)^l_\nu \\ &\quad + \tfrac{1}{2}\bar{\psi}_{\mu i}(g^A_k)_{ij}\nu_5\phi_{\nu j} + \tfrac{1}{2}\bar{\phi}_{\mu i}(g^A_k)_{ij}\nu_5\psi_{\nu j} \end{aligned}$$

4. LAGRANGIAN OF SU(2,2|N) · ⟨N′⟩ SUPERUNIFIED THEORY

Let us construct the Yang-Mills type of Lagrangian of the above $SU(2,2|N)$ supergravity first, that is,

$$\mathscr{L}_{\text{gauge}} = -\tfrac{1}{4}\,\mathrm{St_r}(R_{\mu\gamma}R^{\mu\nu})$$

The μ and ν are tensor indices under the natural frame on M. We can define the metric $g^{\mu\nu}$ on M. Using $g^{\mu\nu}$, we can raise the indices; thus $S_{\text{tr}}(R_{\mu\nu}R^{\mu\nu})$ is a scalar and the $\mathscr{L}_{\text{gauge}}$ is invariant.

Expanding the supertrace $\mathrm{Str}(R_{\mu\nu}R^{\mu\nu})$ into a form corresponding to the generators of supergroup $SU(2,2|N)$, we have

$$\begin{aligned} \mathscr{L}_{\text{gauge}} = -\tfrac{1}{4}[&R^{ab}_{\mu\nu}(M)R^{\mu\nu}_{ab}(M) + R^a_{\mu\nu}(P)R^{\mu\nu}_a(P) + R^a_{\mu\nu}(K)R^{\mu\nu}_a(K) \\ &+ R_{\mu\nu}(D)R^{\mu\nu}(D) + R_{\mu\nu}(A)R^{\mu\nu}(A) + R_{\mu\nu}(S^i)CR^{\mu\nu}(Q^i) \\ &+ R_{\mu\nu}(Q^i)CR^{\mu\nu}(S^i) + R_{\mu\nu}(E_i)R^{\mu\nu}(E_i)] \end{aligned}$$

If we make allowance for the mutual interactions between mass fields and mass fields as well as mass fields and gauge fields, the total Lagrangian of the superunified system become

$$\mathscr{L}_{\text{total}} = \mathscr{L}_{\text{orig}} + \mathscr{L}_{\text{gauge}} + \mathscr{L}'_{\text{inter}}$$

It is evident that the Higgs mechanism has not been introduced into the system at this stage. $\mathscr{L}_{\text{gauge}}$ can be also written as

$$\mathscr{L}_{\text{gauge}} = \mathscr{L}_{\text{gra}} + \mathscr{L}_{\text{inal}} + \mathscr{L}_{\text{sup}} + \mathscr{L}_{\text{ara-sup}} + + \mathscr{L}_{\text{inal-sup}}$$

where

$$\mathscr{L}_{\text{gra}} = -\tfrac{1}{2}R^{AB}_{\mu\nu}(\hat{C})_{\text{gra}}R^{\mu\nu}_{AB}(\hat{C})_{\text{gra}}$$
$$\mathscr{L}_{\text{inal}} = -\tfrac{1}{4}R^{AB}_{\mu\nu}(\bar{I})_{\text{inal}}R^{\mu\nu}_{AB}(\bar{I})_{\text{inal}}$$
$$\mathscr{L}_{\text{sup}} = -\tfrac{1}{4}R^{AB}_{\mu\nu}(S)CR^{\mu\nu}_{AB}(Q) - \tfrac{1}{4}R^{AB}_{\mu\nu}(Q)CR^{\mu\nu}_{AB}(S) - \tfrac{1}{4}R^{AB}_{\mu\nu}(\hat{C})_{\text{sup}}R^{\mu\nu}_{AB}(\hat{C})_{\text{sup}} - \tfrac{1}{4}R^{AB}_{\mu\nu}(\bar{I})_{\text{sup}}R^{\mu\nu}_{AB}(\bar{I})_{\text{sup}}$$
$$\mathscr{L}_{\text{gra-sup}} = -\tfrac{1}{2}R^{AB}_{\mu\nu}(\hat{C})_{\text{gra}}R^{\mu\nu}_{AB}(\hat{C})_{\text{sup}}$$
$$\mathscr{L}_{\text{inal-sup}} = -\tfrac{1}{2}R^{AB}_{\mu\nu}(\bar{I})_{\text{inal}}R^{\mu\nu}_{AB}(\bar{I})_{\text{sup}}$$

That the Lagrangian of the de Sitter gravity

$$\mathscr{L}_{\text{DS}} = -\tfrac{1}{4}F^{ab}_{\mu\nu}F^{\mu\nu}_{ab} - \tfrac{1}{4}V^{ab}_{\mu\nu}V^{\mu\nu}_{ab} - \tfrac{1}{4}J^{a}_{\mu\nu}J^{\mu\nu}_{a} - \tfrac{1}{2}F^{ab}_{\mu\nu}V^{\mu\nu}_{ab}$$

is contained in $\mathscr{L}_{\text{gra}}$ can be seen without difficulty. The first two terms in the above expression are the Einstein term and the cosmology term of DSG, respectively. Thus the superunified theory constructed using the above Lagrangian can contain some gauge theories of gravity as subtheories in it. We know that since the theory of general relativity is a special case of GT_sG, it also can be contained in the superunified theory. For $\mathscr{L}'_{\text{inter}}$ we only have

$$\mathscr{L}'_{\text{inter}} = \mathscr{L}'_{\text{orig-gauge}}$$

Using the replacement

$$\partial_\mu \to D_\mu = \partial_\mu + B^{AB}_\mu \tau_{AB}$$

in $\mathscr{L}_{\text{orig}}$ we can obtain the evolution

$$\mathscr{L}_{\text{orig}} \to \mathscr{L}_{\text{orig}} + \mathscr{L}'_{\text{orig-gauge}}$$

Since we consider only the global invariant spontaneous symmetry breaking of the system here, the mutual interactions between Higgs fields represented in Ω and gauge fields do not appear in $\mathscr{L}'_{\text{inter}}$. That is, $\mathscr{L}_B \subset \mathscr{L}_{\text{orig}}$ cannot undergo evolution as above, and the Goldstone bosons, the number of which is equal to the number of broken generators of group $SU(N)$, will appear in $\mathscr{L}_{\text{orig}}$.

Now we introduce the spontaneous symmetry-breaking Higgs mechanism into the system breaking the group $SU(N)$, i.e.,

$$SU(N) \to SU\langle N'\rangle, \qquad N' \leq N$$

where $SU(N)$ is the usual special unitary $N\times N$ matrix group, $SU\langle N'\rangle$ is either $SU(N')$ or a diagonal matrix subgroup of $SU(N')$, and $SU(N')$ is the usual special unitary $N'\times N'$ matrix group. Then the evolution

$$\mathscr{L}_{\text{B}} \to \mathscr{L}_{\text{B}} + \mathscr{L}_{\text{B-gauge}}$$

will take place. The obtained $\mathscr{L}_{\text{B-gauge}}$ will be contained in the superunified Lagrangian $\mathscr{L}_{\text{su}}$. In addition, according to the GUT, the spontaneous symmetry-breaking Higgs mechanism will bring into the theory new physical couplings. Finally, the following superunified effective Lagrangian can be obtained:

$$\begin{aligned}\mathscr{L}_{\text{su}} &= \mathscr{L}_{\text{orig}} + \mathscr{L}_{\text{gauge}} + \mathscr{L}_{\text{inter}} \\ &= \mathscr{L}_m + \mathscr{L}_{\text{F}} + \mathscr{L}_{\text{B}} + \mathscr{L}_{\text{gauge}} + \mathscr{L}_{\text{orig-gauge}}\end{aligned}$$

As a result, the superspace for supergauge actions will undergo an evolution $U \to \hat{U}$ at this stage, where

$$\hat{U} \equiv V \oplus_{SU(2,2|N)} H \oplus \hat{\Omega}_{SU\langle N' \rangle}$$

Evidently, $\mathscr{L}_{\text{su}}$ still preserves the exact symmetries that are the residual symmetries of the subgroup

$$SU(2,2|N) \cdot \langle N' \rangle \equiv SU(2,2|N) \otimes SU\langle N' \rangle$$

in $\hat{U}$. Thus as far as we know, the structures of the superspace under the gauge supergroup are in close relationship with the structures of the Lagrangian supplied with the superspace.

5. DESCRIPTION OF THE SUPERUNIFIED THEORY USING FIBER BUNDLES

We can give the above supergroup actions on superspace and the evolution theory of the Lagrangian a complete mathematical description using fiber bundle geometry.

Evidently, $\mathscr{L}_{\text{orig}}$ is the action constructed from some geometric invariant quantities under $SU(2,2|N) \cdot (N)$. These quantities are usually supplied by the relative quantities of the globally degenerate trivial principal bundle (Kobayashi and Nomizu, 1963),

$$P'(M, SU(2,2|N) \cdot (N)) \equiv P'(M, SU(2,2|N)) \cup P'(M, SU(N))$$

as well as by a cross section of the associated bundle

$$\begin{aligned}&E'(M, U, SU(2,2|N) \cdot (N), P') \\ &\quad \equiv E'(M, V \otimes H, SU(2,2|N), P') \cup E'(M, \Omega, SU(N), P')\end{aligned}$$

of the bundle P' when the local invariances of the system have completely degenerated into global invariances. We now localize $SU(2,2|N)$, which automatically results in the generating of the principal bundle $P(M, SU(2,2|N))$ and its associated bundle $E(M, V \otimes H, SU(2,2|N), P)$ on the space-time manifold M. Using the geometry of fiber bundles, we can prove that the superconnection on these bundles is just the supergauge potential of the system. Thus we can introduce the horizontal lift basis

$$D_\mu = \partial_\mu + B_\mu^{AB}\tau_{AB}$$

into these bundle spaces in terms of the superconnections B_μ^{AB} and with the Lie product of these bases can obtain the supercurvature $R_{\mu\nu}^{AB}$ on these bundles as

$$[D_\mu, D_\nu] = R_{\mu\nu}^{AB}\tau_{AB}$$

It can also be proved that the supercurvature $R_{\mu\nu}^{AB}$ is the supergauge strength of the system. Thus the supergauge field quantities B_μ^{AB} and $R_{\mu\nu}^{AB}$ of the superunified theory can be provided with the geometric quantities belonging to the bundle, $P(M, SU(2,2|N))$ or $E(M, V\otimes H, SU(2,2|N, P)$, and $\mathscr{L}_{\text{gauge}}$, which is the localization result of the supergauge symmetries, can be provided by the invariant quartic form of the supercurvature scalar on the bundles. The replacement

$$\partial_\mu \to D_\mu = \partial_\mu + B_\mu^{AB}\tau_{aB}$$

in the Lagrangian is just the replacement of the coordinate basis by the noncoordinate basis D_μ on these bundles. Since the group $SU(N)$ has not been localized, if we use $SU(2,2|N)\cdot(N)$ as the structure group, we can only establish the following local degenerate bundles:

$$\dot{P}(M) = \dot{P}(M, SU(2,2|N)\cdot(N)) \equiv P(M, SU(2,2|N)) \cup P'(M, SU(N))$$

and

$$\begin{aligned}\dot{E}(M) &= \dot{E}(M, U, SU(2,2|N)\cdot(N), \dot{P})\\ &\equiv E(M, V\otimes H, SU(2,2|N), P) \cup E'(M, \Omega, SU(N), P')\end{aligned}$$

From the geometric quantities on the above bundles $\dot{P}(M)$ or $\dot{E}(M)$, the Lagrangian

$$\mathscr{L}_{\text{total}} = \mathscr{L}_{\text{orig}} + \mathscr{L}_{\text{gauge}} + \mathscr{L}'_{\text{inter}}$$

can be obtained.

To introduce the Higgs mechanism into the system, the group $SU(N)$ needs to be localized, and the bundles $P(M, SU(N))$ and $E(M, \Omega, SU(N), P)$ can be generated as a result. Carrying out a procedure similar to the above and using the replacement

$$\partial_\mu \to D_\mu = \partial_\mu + (E^i_{\ \mu} g_i + A_\mu A)$$

we can achieve the evolution

$$\mathscr{L}_{\mathrm{B}} \to \mathscr{L}_{\mathrm{B}} + \mathscr{L}_{\mathrm{B\text{-}gauge}}$$

Since the supergroup $SU(2,2|N)\cdot(N)$ has been completely localized, the bundles

$$\begin{aligned} P(M) &= P(M, SU(2,2|N)\cdot(N)) \\ &\equiv P(M, SU(2,2|N)) \cup P(M, SU(N)) \end{aligned}$$

and

$$\begin{aligned} E(M) &= E(M, U, SU(2,2|N)\cdot(N)) \\ &\equiv E(M, V\oplus H, SU(2,2|N), P) \cup E(M, \Omega, SU(N), P) \end{aligned}$$

can be obtained at this stage. The cross section $\sigma_E(x)$ of bundle $E(M)$ is precisely the differentiable distribution of a set of physical quantities that are in U and on M and invariant under the supergroup $SU(2,2|N)\cdot(N)$. The cross section $\sigma_p(\pi)$ on the bundle $P(M)$ is just the differentiable distribution of a supervierbein that is on M and under $SU(2,2|N)\cdot(N)$ and relevant to the differentiable distribution of the above physical quantities.

Since there is spontaneous symmetry breaking in the system, the Higgs mechanism needs to be introduced into it now, and we have proved that the Higgs mechanism of the theory is equivalent to the reduction of the bundle $P(M)$ or $E(M)$ describing the theory. The supercurvature scalar and other geometric invariant quantities which belong to the obtained reduction subbundle of $P(M)$ or $E(M)$ can be utilized to construct the $\mathscr{L}_{\mathrm{su}}$. The Higgs mechanism can be completely described by fiber bundle geometry.

As $SU(N)\to SU\langle N'\rangle$ we perform the reductions of these bundles:

$$\begin{aligned} P(M)\to \hat{P}(M) &= \hat{P}(M, SU(2,2|N)\cdot\langle N'\rangle) \\ &\equiv P(M, SU(2,2|N)) \cup \hat{P}(M, SU\langle N'\rangle) \end{aligned}$$

and

$$\begin{aligned} E(M)\to \hat{E}(M) = \hat{E}(M) &= \hat{E}(M, U, SU(2,2|N)\cdot\langle N'\rangle, \hat{P}) \\ &\equiv E(M, V\oplus H, SU(2,2|N), P) \cup E(M, \hat{\Omega}, SU\langle N'\rangle, P) \end{aligned}$$

Then $\mathscr{L}_{\mathrm{su}}$ is provided by just the quartic form of the supercurvature scalar and other geometric invariant quantities on the subbundle $\hat{E}(M)$ or $\hat{P}(M)$.

The evolution procedure of the supergauge symmetry properties of the unified system and its Lagrangian can be written as follows:

Physics:	Superunified system	→	Supergauge symmetries	→
Mathematics:	Superspace		Supergroup	
	$U = V \oplus_{SU(2,2\|N)} H \oplus \Omega_{SU(N)}$	→	$SU(2,2\|N)$	→
Langrangian:	$\mathscr{L}_{\text{orig}} = \mathscr{L}_m + \mathscr{L}_F + \mathscr{L}_B$	→		
	Localization of the supergauge symmetries	→	Higgs mechanism	→
	Bundles		Reduction of bundles	
	$\dot{P}(M), \dot{E}(M)$	→	$P(M) \to \hat{P}(M)$, $E(M) \to \hat{E}(M)$	→
	$\mathscr{L}_{\text{total}} = \mathscr{L}_{\text{orig}} + \mathscr{L}_{\text{gauge}} + \mathscr{L}'_{\text{inter}}$	→		
	Local exact symmetries of the system			
	Reduction subbundles $\hat{E}(M), \hat{P}(M)$			
	$\mathscr{L}_{\text{Su}} = \mathscr{L}_{\text{orig}} + \mathscr{L}_{\text{gauge}} + \mathscr{L}_{\text{inter}}$			

Finally, we arrive at the conclusion that on a suitable superspace the Lagrangian of any system possessing the action mechanism under a relevant supergroup, as above mentioned, can be defined in terms of definite evolutions of the geometric quantities on a fiber bundle in an exact way. Therefore, the fiber bundle geometry can give the superunified theory a clear, definite, and complete mathematical description.

REFERENCES

Bergshoeff, E., de Roo, M., van Holten, J. W., de Wit, B., and van Proeyen, A. (1980). Extended conformal supergravity and its applications.

Beyl, L. M. (1979). *Physical Review*, **19**, 1732.

Ferrara, S., Kaku, M., Townsend, P. K., and Van Nieuwenhuizen, P. (1977). *Nuclear Physics*, B**129**, 125.

Ferrara, S., Grisaru, M. T., and van Nieuwenhuizen, P. (1978). *Nuclear Physics*, **B138**, 430.

Kaku, M., Townsend P. K., and Van Nieuwenhuizen, P. (1977a). *Physical Review Letters*, **39**, 1109.

Kaku, M., Townsend, P. K., and van Nieuwenhuizen, P. (1976b). *Physics Letters*, **68B**, 304.

Kaku, M., Townsend, P. K., and van Nieuwenhuizen, P. (1978). *Physical Review*, **D17**, 3179.

Kobahashi, S., and Nomizu, K. (1963). *Foundations of Differential Geometry*, Vol. I.

Langacker, P. (1981). *Physics Reports*, **72**, (4).

Shao Changgui. The embedding of grand unified theories in the $SU(2,2|N)\langle N' \rangle$ superunified theory.